Materials to support Able Mathematicians
in
Key Stage 3

Introduction / rationale

The materials have been written and collated by Cumbria's Leading Mathematics Teachers.

The materials are designed to provide support for Able Key Stage 3 Mathematicians. There are a range of activities offered including individual questions that could be used in starters, investigations that could build upon work being undertaken in the classroom and longer tasks requiring use of a range of skills and knowledge.

There is a wealth of material already available and some of this is referenced, the list is not exhaustive and is only intended to highlight some of the materials already existing.

The index of this document is hyperlinked to the relevant sections.

Index

National Curriculum Levels

Ma 1: Using and applying mathematics

Level 7

Starting from problems or contexts that have been presented to them, pupils progressively refine or extend the mathematics used to generate fuller solutions. They give a reason for their choice of mathematical presentation, explaining features they have selected. Pupils justify their generalisations, arguments or solutions, showing some insight into the mathematical structure of the problem. They appreciate the difference between mathematical explanation and experimental evidence.

Level 8

Pupils develop and follow alternative approaches. They reflect on their own lines of enquiry when exploring mathematical tasks; in doing so they introduce and use a range of mathematical techniques. Pupils convey mathematical or statistical meaning through precise and consistent use of symbols that is sustained throughout the work. They examine generalisations or solutions reached in an activity, commenting constructively on the reasoning and logic or the process employed, or the results obtained, and make further progress in the activity as a result.

Exceptional performance

Pupils give reasons for the choices they make when investigating within mathematics itself or when using mathematics to analyse tasks; these reasons explain why particular lines of enquiry or procedures are followed and others rejected. Pupils apply the mathematics they know in familiar and unfamiliar contexts. Pupils use mathematical language and symbols effectively in presenting a convincing reasoned argument. Their reports include mathematical justifications, explaining their solutions to problems involving a number of features or variables.

Ma 2: Number and algebra

Level 7

In making estimates, pupils round to one significant figure and multiply and divide mentally. They understand the effects of multiplying and dividing by numbers between 0 and 1. Pupils solve numerical problems involving multiplication and division with numbers of any size, using a calculator efficiently and appropriately. They understand and use proportional changes, calculating the result of any proportional change using only multiplicative methods. Pupils find and describe in symbols the next term or nth term of a sequence where the rule is quadratic; they multiply two expressions of the form $(x + n)$; they simplify the corresponding quadratic expressions. Pupils use algebraic and graphical methods to solve simultaneous linear equations in two variables. They solve simple inequalities.

Level 8

Pupils solve problems involving calculating with powers, roots and numbers expressed in standard form, checking for correct order of magnitude. They choose to use fractions or percentages to solve problems involving repeated proportional changes or the calculation of the original quantity given the result of a proportional change. They evaluate algebraic formulae, substituting fractions, decimals and negative numbers. They calculate one variable, given the others, in formulae such as V = Yr2h. Pupils manipulate algebraic formulae, equations and expressions, finding common factors and multiplying two linear expressions. They know that a 2 -b 2= (a+b)(a - b). They solve inequalities in two variables. Pupils sketch and interpret graphs of linear, quadratic, cubic and reciprocal functions, and graphs that model real situations.

Exceptional performance

Pupils understand and use rational and irrational numbers. They determine the bounds of intervals. Pupils understand and use direct and inverse proportion. In simplifying algebraic expressions, they use rules of indices for negative and fractional values. In finding formulae that approximately connect data, pupils express general laws in symbolic form. They solve simultaneous equations in two variables where one equation is linear and the other is quadratic. They solve problems using intersections and gradients of graphs.

Ma 3: Shape, space and measures

Level 7

Pupils understand and apply Pythagoras' theorem when solving problems in two dimensions. They calculate lengths, areas and volumes in plane shapes and right prisms. Pupils enlarge shapes by a fractional scale factor, and appreciate the similarity of the resulting shapes. They determine the locus of an object moving according to a rule. Pupils appreciate the imprecision of measurement and recognise that a measurement given to the nearest whole number may be inaccurate by up to one half in either direction. They understand and use compound measures, such as speed.

Level 8

Pupils understand and use congruence and mathematical similarity. They use sine, cosine and tangent in right-angled triangles when solving problems in two dimensions. They distinguish between formulae for perimeter, area and volume, by considering dimensions.

Exceptional performance

Pupils sketch the graphs of sine, cosine and tangent functions for any angle, and generate and interpret graphs based on these functions. Pupils use sine, cosine and tangent of angles of any size, and Pythagoras' theorem when solving problems in two and three dimensions. They use the conditions for congruent triangles in formal geometric proofs [for example, to prove that the base angles of an isosceles triangle are equal]. They calculate lengths of circular arcs and areas of sectors, and calculate the surface area of cylinders and volumes of cones and spheres. Pupils appreciate the continuous nature of scales that are used to make measurements.

Ma 4: Handling data

Level 7

Pupils specify hypotheses and test them by designing and using appropriate methods that take account of variability or bias. They determine the modal class and estimate the mean, median and range of sets of grouped data, selecting the statistic most appropriate to their line of enquiry. They use measures of average and range, with associated frequency polygons, as appropriate, to compare distributions and make inferences. They draw a line of best fit on a scatter diagram, by inspection. Pupils understand relative frequency as an estimate of probability and use this to compare outcomes of experiments.

Level 8

Pupils interpret and construct cumulative frequency tables and diagrams, using the upper boundary of the class interval. They estimate the median and interquartile range and use these to compare distributions and make inferences. They understand how to calculate the probability of a compound event and use this in solving problems.

Exceptional performance

Pupils interpret and construct histograms. They understand how different methods of sampling and different sample sizes may affect the reliability of conclusions drawn. They select and justify a sample and method to investigate a population. They recognise when and how to work with probabilities associated with independent mutually exclusive events.

The Framework Objectives
Year 9 Able

Using and applying mathematics

Applying mathematics and solving problems

2-25 Generate fuller solutions to problems.
30-1 Recognise limitations on accuracy of data and measurements; give reasons for choice of presentation, explaining selected features and showing insight into the problem's structure.
32-3 Justify generalisations, arguments or solutions; pose extra constraints and investigate whether particular cases can be generalised further.

Numbers and the number system

Place value, ordering and rounding

36-9 Write numbers in standard form.
44-7 Understand upper and lower bounds; round numbers to three decimal places and a given number of significant figures.

Integers, powers and roots

58-9 **Know and use the index laws for multiplication and division of positive integer powers; begin to extend understanding of index notation to negative and fractional powers**, recognising that the index laws can be applied to these as well.

Fractions, decimals, percentages, ratio and proportion

64-5 Use algebraic methods to convert a recurring decimal to a fraction in simple cases.
78-81 **Understand and use proportionality and calculate the result of any proportional change using multiplicative methods**; understand the implications of enlargement for area and volume.

Calculations

Number operations and the relationships between them

82-3 Recognise and use reciprocals.

Mental methods and rapid recall of number facts

102-7 Estimate calculations by rounding numbers to one significant figure and multiplying or dividing mentally.

Calculator methods

108-9 Use a calculator efficiently and appropriately, including using the reciprocal key and entering and interpreting numbers in standard form.

Algebra

Equations, formulae and identities

114-5 Know and use the index laws in generalised form for multiplication and division of positive integer powers.
118-21 Square a linear expression, expand the product of two linear expressions of the form x ± n and simplify the corresponding quadratic expression; **establish identities** such as a^2 ? b^2 = $(a + b)(a - b)$.
126-9 **Solve a pair of simultaneous linear equations by eliminating one variable; link a graphical representation of an equation or a pair of equations to the algebraic solution**; consider cases that have no solution or an infinite number of solutions.
130-1 Solve linear inequalities in one variable, and represent the solution set on a number line; begin to solve inequalities in two variables.
138-41 Derive and use more complex formulae, and **change the subject of a formula**.

Sequences, functions and graphs

148-53 Find the next term and the nth term of quadratic sequences and functions and explore their properties.
158-9 Deduce properties of the sequences of triangular and square numbers from spatial patterns.
162-3 Plot the graph of the inverse of a linear function; know simple properties of quadratic functions.
168-9 Investigate the gradients of parallel lines and lines perpendicular to these lines.
170-1 Plot graphs of simple quadratic and cubic functions, e.g. $y = x^2$, $y = 3x^2 + 4$, $y = x^3$.

Shape, space and measures

Geometrical reasoning: lines, angles and shapes

178-9 Distinguish between practical demonstration and proof; know underlying assumptions, recognising their importance and limitations and the effect of varying them.
186-9 **Understand and apply Pythagoras' theorem.**
190-1 **Apply the conditions SSS, SAS, ASA or RHS to establish the congruence of triangles.**
192-3 **Know that if two 2-D shapes are similar, corresponding angles are equal and corresponding sides are in the same ratio.**
196-7 Know that the tangent at any point on a circle is perpendicular to the radius at that point; explain why the perpendicular from the centre to the chord bisects the chord.

Transformations

202-17 Enlarge 2-D shapes, given a fractional scale factor; recognise the similarity of the resulting shapes; understand the implications of enlargement for area and volume.

Coordinates

218-19 Find points that divide a line in a given ratio, using the properties of similar triangles; given the coordinates of points A and B, calculate the length of AB.

Construction and loci

222-3 **Know from experience of constructing them that triangles given SSS, SAS, ASA or RHS are unique, but that triangles given SSA or AAA are not.**
224-7 Find the locus of a point that moves according to a more complex rule involving loci and simple constructions, both by reasoning and by using ICT.

Measures and mensuration

230-1 Recognise that measurements given to the nearest whole unit may be inaccurate by up to one half of the unit in either direction.
232-3 **Understand and use measures of speed (and other compound measures such as density or pressure) to solve problems;** solve problems involving constant or average rates of change.
234-7 Know and use the formulae for length of arcs and area of sectors of circles.
238-41 Calculate lengths, areas and volumes in right prisms, including cylinders.
242-7 Begin to use sine, cosine and tangent in right-angled triangles to solve problems in two dimensions.

Handling data

Specifying a problem, planning and collecting data

250-1 Identify possible sources of bias and plan how to minimise it.
254-5 Identify what extra information may be required to pursue a further line of enquiry.

Processing and representing data, using ICT as appropriate

256-61 Find the median and quartiles for large data sets; estimate the mean, median and interquartile range of a large set of grouped data.
262-7 Select, construct and modify, on paper and using ICT, suitable graphical representation to progress an enquiry, including:

- frequency polygons
- lines of best fit by eye, understanding what they represent

Identify key features present in the data.

Interpreting and discussing results

270-1 Analyse data to find patterns and exceptions, look for cause and effect and try to explain anomalies.
274-5 **Examine critically the results of a statistical enquiry, and justify choice of statistical representation in written presentations**, recognising the limitations of any assumptions and their effect on conclusions drawn.

Probability

282-3 Understand relative frequency as an estimate of probability and use this to compare outcomes of experiments.

Starter Questions

It is suggested that these questions can be regularly used with the most able students.

The first section are linked to Prime Numbers, the second linked to Percentages.

Mainly Connected with Primes

Last digit

E.g.
(a) $58239 \times 1702 \times 37$
(b) $(58)^3$
(c) $12!$
(d) 3^{333}
(e) $(**)^3 = ****3$ all possible solutions
(f) $(**)^2 = 33*4$

Prime Factorisation

Find largest prime factor of 171250 say.

- How many factors does 171250 have? → Which numbers have an odd number of factors? Explain.
- What is the smallest integer you would need to multiply by to make 171250 into a square/cube?
- HCF (171250, 93500)
- LCM (171250, 93500)

Q1. A blue lamp flashes every 4992 seconds
A green lamp flashes every 30250 seconds
A yellow lamp flashes every 1377 seconds
If all 3 lamps flashed simultaneously when you were born, is it likely that you would still be alive to see it happen again?

Q2. Arrange the digits 1, ..., 9 to form a 9 digit number that is divisible by 1, 2, ..., 8, 9.

Q3. Smallest number that is divisible by 1, ..., 7.

Q4. Smallest number that is divisible by 7 but has remainder 1 when divided by 1, ..., 6.

Q5. Is the number *xxyy* a perfect square for any *x*, *y* ? 0, 1, ..., 9.

Q6. The product of the ages of 3 secondary school children is 1848. How old is the eldest?

Q7. $\sqrt{17424}$ No calculators

Q8. 100! Ends with how many zeros?

Q9. What is the smallest value of *n* for which $\frac{n}{47520}$ has a decimal expansion which terminates?

Excellent questions in Mathematical Puzzling, Chapters 6, 24, 28:	*Prime time: one* *Prime time: two* *Prime time: three*

Mainly Connected with Primes Answers

Last digit

E.g.
- (a) 58239 x 1702 x 37 **Answer = 6**
- (g) $(58)^3$ **Answer = 2**
- (h) 12! **Answer = 0**
- (i) 3^{333} **Answer = 3** (Need to explore the patterns of the last digit starting from 3^1)
- (j) $(**)^3$ = ****3 **Answer = 27, 37**
- (k) $(**)^2$ = 33*4 **Answer = $(58)^2$ = 3364**

Prime Factorisation

Find largest prime factor of 171250 say. **Answer = 137 ($2 \times 5^4 \times 137$)**

- How many factors does 171250 have? **Answer = 20 factors** → Which numbers have an odd number of factors? Explain. **Answer = square numbers, the square root appears once.**
- What is the smallest integer you would need to multiply by to make 171250 into a square/cube? **Answer = 274 / 1876900**
- HCF (171250, 93500) **Answer = 250**
- LCM (171250, 93500) **Answer = 64047500**

Q1. A blue lamp flashes every 4992 seconds
A green lamp flashes every 30250 seconds
A yellow lamp flashes every 1377 seconds
If all 3 lamps flashed simultaneously when you were born, is it likely that you would still be alive to see it happen again? **Answer = no, 1098 years (using $365^1/_4$ days per year)**

Q2. Arrange the digits 1, ..., 9 to form a 9 digit number that is divisible by 1, 2, ..., 8, 9. **Answer = you can't to be divisible by 5 it must end in 0 or 5, 0 not available so must end in 5. To be divisible by 8 must end in 0, 2, 4, 6, 8 therefore not possible.**

Q3. Smallest number that is divisible by 1, ..., 7. **Answer = 420**

Q4. Smallest number that is divisible by 7 but has remainder 1 when divided by 1, ..., 6. **Answer = dividing by 1 never leaves a remainder so no solution.**

Q5. Is the number *xxyy* a perfect square for any *x*, *y* ? 0, 1, ..., 9. **Answer = 7744 = 88^2**

Q6. The product of the ages of 3 secondary school children is 1848. How old is the eldest? **Answer = 11, 12, 14 so 14**

Q7. $\sqrt{17424}$ No calculators **Answer = $17424 = 2^4 \times 3^2 \times 11^2$ so $\sqrt{17424} = 2^2 \times 3^1 \times 11^1 = 132$**

Q8. 100! Ends with how many zeros? **Answer = 21**

Q9. What is the smallest value of n for which $\frac{n}{47520}$ has a decimal expansion which terminates?
Answer = 297

Problem Percentages

1.a) What is 10% of 10 plus 20% of 20 plus 30% of 30… plus 100% of 100?

b) What is 100% of 10 plus 90% of 20 plus 80% of 30… plus 10% of 100?

2.a) In a football tournament, 25% of games were goalless, 35% recorded only 1 goal, 12 games had 2 goals in them, 3 games had 3 goals and 1 game recorded 6 goals. How many goals were scored in total at the tournament?

b) In a cricket league, during a season, 7.5% of games were tied, 65% of games ended in a result and the rest finished as a draw. What is the minimum number of teams in the league?

3. 20% of 30% of 40% of 50% of x is a positive whole number. What is the smallest value x can take? (Could break down – 20% of x, 20% of 30% of x, etc.)

4. After the first 3 races at the track, Sid had lost 40% of his money. However, over the rest of the evening he won back 40% of what he had lost. What percentage of his money did he lose over the whole evening.

5. Last year I earned 30% more than Frank. We both had a wage rise of £1.25. Now I earn 24% more than Frank. What is my current wage?

6. At 6am every morning I chop 20% off the height of a plant, but by sunset each day its height has increased by 20%. After how many days will the plant be less than half its original size?

7. At a supermarket, I have a voucher which will allow me to get 10% off my weekly shop. I also work at the supermarket, which means I automatically get 25% off. Should I hand over my voucher before or after my 25% gets knocked off? Why?

8. The numbers at this year's (2005) competition are 30% down on last year but 6% up on the numbers 2 years ago. What was the % increase in the numbers from 2003 to 2004?

9. Find a 2-digit such that when its digits are reversed it decreases by 24%.

10. A cuboid has dimensions 55 x 70 x 90. One of its dimensions increases by 10%, another by 20%, and the other stays the same.

a) What is the maximum % increase in the volume and how is this achieved?

b) What is the maximum % increase in the surface area and how is this achieved?

Problem Percentages Answers

1.a) What is 10% of 10 plus 20% of 20 plus 30% of 30… plus 100% of 100? **Answer = 385**

b) What is 100% of 10 plus 90% of 20 plus 80% of 30… plus 10% of 100? **Answer = 110**

2.a) In a football tournament, 25% of games were goalless, 35% recorded only 1 goal, 12 games had 2 goals in them, 3 games had 3 goals and 1 game recorded 6 goals. How many goals were scored in total at the tournament?
Answer = 40 games in total, 14 1 goal +12, 2 goals + 3, 3 goals + 1, 6 goals = 53 goals

b) In a cricket league, during a season, 7.5% of games were tied, 65% of games ended in a result and the rest finished as a draw. What is the minimum number of teams in the league?
Answer = 40 games

3. 20% of 30% of 40% of 50% of *x* is a positive whole number. What is the smallest value *x* can take? (Could break down – 20% of *x*, 20% of 30% of *x*, etc.)
Answer = 250

4. After the first 3 races at the track, Sid had lost 40% of his money. However, over the rest of the evening he won back 40% of what he had lost. What percentage of his money did he lose over the whole evening.
Answer = 24%

5. Last year I earned 30% more than Frank. We both had a wage rise of £1.25. Now I earn 24% more than Frank. What is my current wage? **Answer = £7.75**

6. At 6am every morning I chop 20% off the height of a plant, but by sunset each day its height has increased by 20%. After how many days will the plant be less than half its original size? **Answer = 17 days**

7. At a supermarket, I have a voucher which will allow me to get 10% off my weekly shop. I also work at the supermarket, which means I automatically get 25% off. Should I hand over my voucher before or after my 25% gets knocked off? Why?
Answer = either way, it makes no difference as you are multiplying

8. The numbers at this year's (2005) competition are 30% down on last year but 6% up on the numbers 2 years ago. What was the % increase in the numbers from 2003 to 2004? **Answer = 51.4%**

9. Find a 2-digit such that when its digits are reversed it decreases by 24%.
Answer = 75 (75 and 57)

10. A cuboid has dimensions 55 x 70 x 90. One of its dimensions increases by 10%, another by 20%, and the other stays the same.

a) What is the maximum % increase in the volume and how is this achieved?
Answer = 32% increase in volume, the order doesn't matter

b) What is the maximum % increase in the surface area and how is this achieved?
Answer = 22.5% increase the 90 by 20% and the 70 by 10%

Egyptian Fractions

The Ancient Egyptians used **unit** fractions.

These are fractions with 1 as the numerator, such as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ and so on.

To make other fractions, the Egyptians added unit fractions:

eg. $\frac{3}{4} = \frac{2}{4} + \frac{1}{4} = \frac{1}{2} + \frac{1}{4}$

Unit fractions could not be repeated, so $\frac{2}{5} = \frac{1}{5} + \frac{1}{5}$ is not allowed.

1. Show that:

   (a) $\frac{3}{5} = \frac{1}{2} + \frac{1}{10}$ (b) $\frac{3}{7} = \frac{1}{3} + \frac{1}{11} + \frac{1}{231}$

2. Write fractions with 2 as the denominator using Egyptian fractions,

   eg. $\frac{2}{5}$, $\frac{2}{7}$, $\frac{2}{9}$, …

   Describe any patterns you notice in your solutions.

3. Can you find a rule for writing fractions of the form $\frac{2}{n}$ using Egyptian fractions?

4. Investigate writing other fractions as Egyptian fractions.

Why do you think the Ancient Egyptians used this method for working with fractions?

What are the advantages and disadvantages of Egyptian fractions?

Egyptian Fractions References

National Curriculum Levels

Ma1: Using and applying mathematics

Level 7
Level 8
Exceptional Performance

Ma2: Number and algebra

Level 8
Pupils manipulate algebraic formulae, equations and expressions, finding common factors

Numeracy Framework

Year 9
Pages 60-81

Egyptian Fractions – Teacher Notes

1. (a) $\frac{3}{5} = \frac{6}{10} = \frac{5}{10}+\frac{1}{10} = \frac{1}{2}+\frac{1}{10}$

   (b) $\frac{3}{7} = \frac{9}{21} = \frac{7}{21}+\frac{2}{21} = \frac{1}{3}+\frac{22}{231} = \frac{1}{3}+\frac{21}{231}+\frac{1}{231} = \frac{1}{3}+\frac{1}{11}+\frac{1}{231}$

2. $\frac{2}{5}=\frac{1}{3}+\frac{1}{15}$, $\frac{2}{7}=\frac{1}{4}+\frac{1}{28}$, $\frac{2}{9}=\frac{1}{5}+\frac{1}{45}$, etc.

   The denominator of the larger unit fraction is equal to half the denominator of the original fraction, rounded up to the nearest whole number.

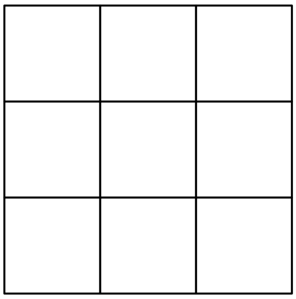
   The product of the denominator of the original fraction and the larger unit fraction is the denominator of the smaller unit fraction.

3. If the numerator is 2 then the denominator can be written in the form 2k+1.

$$\frac{2}{2k+1}=\frac{2(k+1)}{(2k+1)(k+1)}=\frac{(2k+1)+1}{(2k+1)(k+1)}$$
$$=\frac{(2k+1)}{(2k+1)(k+1)}+\frac{1}{(2k+1)(k+1)}$$
$$=\frac{1}{k+1}+\frac{1}{(2k+1)(k+1)}$$

4. Fractions of the form $\frac{3}{n}$ can be considered, looking at fractions of the forms $\frac{3}{3k+1}$ and $\frac{3}{3k+2}$, leading to further generalisation.

Counting Squares



1. How many squares can you see in the grid above?
   Can you see more than 9?

   To help you solve the problem, break it down into smaller tasks:

      How many 1×1 squares are there?
      How many 2×2 squares are there?
      How many 3×3 squares are there?

2. How many squares would you find in a 4×4 grid?

3. Extend the problem by looking at other sized grids.

   Can you find a way to work out the number of squares in any grid?

4. How many rectangles can you see in the original 3×3 grid?

Counting Squares – Teacher Notes

1.

Size of square	Number of squares
1×1	9
2×2	4
3×3	1

In a 3×3 grid there are $1 + 4 + 9 = 14$ squares

2.

Size of square	Number of squares
1×1	16
2×2	9
3×3	4
4×4	1

In a 4×4 grid there are $1 + 4 + 9 + 16 = 30$ squares

3. In the trivial case, there is 1 square in a 1×1 grid.
For 2×2, there are 5 squares.

In general, the number of squares in an $n \times n$ grid is given by the sum of the square numbers up to n^2.

The formula for this sum is $\frac{1}{6}n(n+1)(2n+1)$.

4. For non-square rectangles:

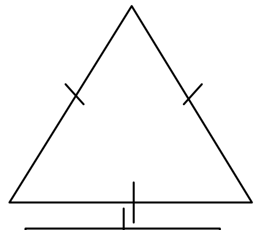
Size of rectangle	Number of rectangles
1×2 or 2×1	12
1×3 or 3×1	6
2×3 or 3×2	4

In a 3×3 grid there are 22 non-square rectangles in addition to the 14 squares, giving a total of $36 = 6^2$.

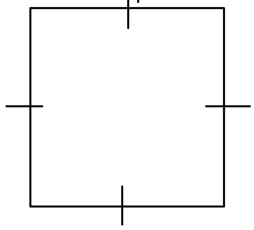
In general, the number of rectangles in an $n \times n$ grid is the nth triangle number squared, $\left(\frac{1}{2}n(n+1)\right)^2 = \frac{1}{4}n^2(n+1)^2$

Area of a Regular Polygon

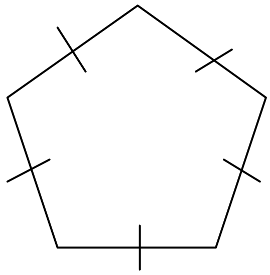
Task 1: Calculate the area of an equilateral triangle with side length n.



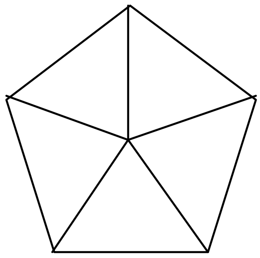
Task 2: Calculate the area of a square of side length n.



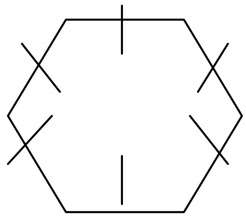
Task 3: Calculate the area of a regular pentagon of side length n.



Hint: You will need to split it into triangles first, as shown below.



Task 4: Calculate the area of a regular hexagon of side length n. You will need to split it into triangles as you did with the pentagon.



Task 5: Look at your results and try to deduce a formula for the area of an N- sided regular polygon. You may need to re-visit Tasks 1 and 2 and find a formula for a triangle and a square by splitting them into triangles as you did with the pentagon and hexagon. Can you convince yourself that your new formulae are the same as the ones you started with?

Task 6: Check that your final formula works by using it to predict the area of a octagon.

Area of a Regular Polygon Teachers Notes

In order to tackle this piece of work, pupils should have already met the following:

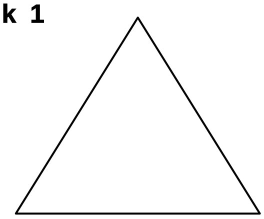
Area of triangles
Trigonometry
Angles in regular polygons

This piece of work is designed to be used with small groups so that pupils are able to discuss ideas and justify results to each other. Although teacher input should not be necessary to start the task they will benefit from the opportunity to discuss the work in progress and to draw out what they have discovered.

Having completed this task, pupils are then in a position to progress to the **QCA Awkward Areas task.**

A brief solution is outlined below. Note that it is possible to work with the angle at the centre rather than the angle at the edge.

Task 1



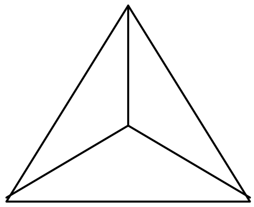
The obvious solution for the area is

$$\frac{1}{2}n^2\sin 60^\circ$$

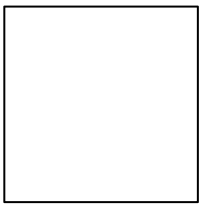
however considering the triangle on the left gives the formula

$$3\times \frac{1}{2}n\times \frac{1}{2}n\tan 30^\circ = \frac{3}{4}n^2\tan 30^\circ$$

which a calculator should show is the same.



Task 2



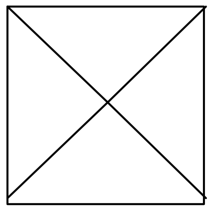
The obvious solution for the area is

$$n^2$$

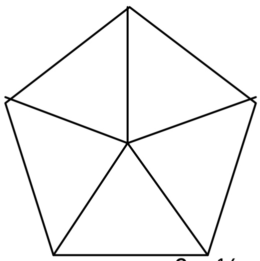
however considering the square on the left gives the formula

$$4\times \frac{1}{2}n\times \frac{1}{2}n\tan 45^\circ = n^2\tan 45^\circ$$

which a calculator should show is the same.



Task 3



Pupils will need to work out the angle at the centre for each of the triangles and hence the angles at the edge. This gives the formula

$$5\times \frac{1}{2}n\times \frac{1}{2}n\tan 54^\circ = 5\times \frac{1}{4}n^2\tan 54^\circ$$

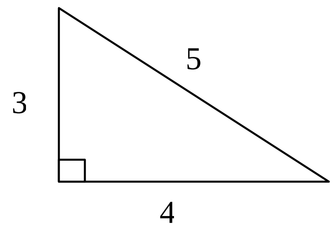
Task 4 $6\times \frac{1}{2}n\times \frac{1}{2}n\tan 60^\circ = 6\times \frac{1}{4}n^2\tan 60^\circ$

Task 5 Area of an N sided polygon is $N\times \frac{1}{4}n^2\tan\left(\frac{1}{2}(180 - 360/N)\right)^\circ$

or $N\times \frac{1}{4}n^2\tan(90 - 180/N)^\circ$

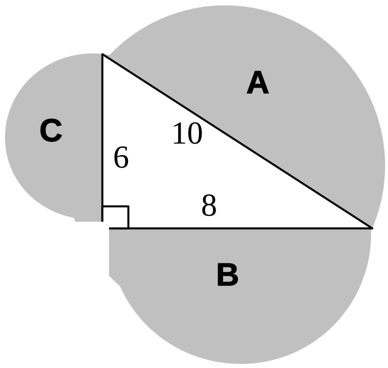
Beyond Pythagoras

Task A Show how Pythagoras' Theorem works for the following triangle.



What if we use different shapes instead of squares on each edge? Does Pythagoras' Theorem still work?

Task B Using semicircles:

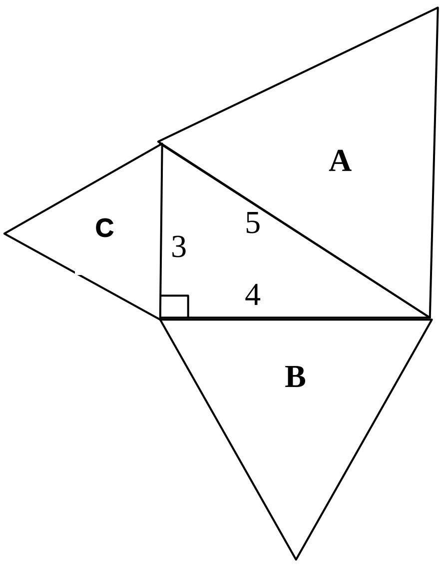


(i) Calculate the areas of the semi-circles labelled A, B, C.

(ii) Does A = B + C?

(iii) Is Pythagoras' Theorem still true with semi-circles?

Task C Using equilateral triangles:



(i) Calculate the areas of the triangles labelled A, B, C.

(ii) Does A = B + C?

(iii) Is Pythagoras' Theorem still true with equilateral triangles?

Task D Now try using different regular shapes.
Does Pythagoras' Theorem work for any regular shape?

Task E Now try using any 2D shape.
Does Pythagoras' Theorem work for any 2D shape?

Beyond Pythagoras Teachers Notes

In order to tackle this piece of work, pupils should have already met the following:

Pythagoras' Theorem
Using trigonometry in right-angled triangles
Use the formula for area of a circle
(Construction of regular 2D shapes)

This piece of work is designed to be used with small groups so that pupils are able to discuss ideas and justify results to each other. Although teacher input should not be necessary to start the task they will benefit from the opportunity to discuss the work in progress and to draw out what they have discovered.

Pupils can be asked to use compasses and a straight edge to construct the shapes on each edge of the triangle.

A brief solution is outlined below.

Task B (i) Area A = ½.p.5^2 = 12½p
Area B = ½.p.3^2 = 4½p
Area C = ½.p.4^2 = 8p
Discuss the use of leaving answers in terms of p for accuracy rather than using decimals.

(ii) Yes because 12½p + 4½p + 8p

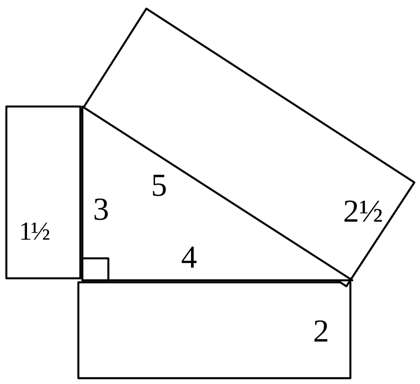
(iii) Yes because Area A = Area B + Area C – remind pupils that Pythagoras' Theorem is based on area.

Task C (i) Encourage pupils to try and construct the shape using compasses

(ii) Discuss various ways of measuring area of the triangles, eg:
counting squares
measuring the height of the triangle
using Pythagoras' theorem to find the height of the triangle
using trigonometry to find the height of the triangle
using the area formula ½bc sinA

Task D Encourage use of pentagons, hexagons, etc – pupils may need to use trigonometry to find the heights and hence the areas.

Task E Encourage use of the special quadrilaterals and isosceles triangles before they move on to other bizarre shapes.
Once they decide that they do not obey Pythagoras' Theorem, push them towards similar shapes eg rectangles with the width half the length.



(i) Area A = 5x2½ = 12½
Area B = 4x2 = 8
Area C = 3x1½ = 4½

(ii) 8 + 4½ = 12½
ie Area A = Area B + Area C
so Pythagoras' Theorem does work for similar 2D shapes of which regular polygons are a special case.

The Long Tunnel

Many of the tunnels in the Alps have a minimum and maximum speed limit for the length of the tunnel.

The limits are there for safety purposes so that as many cars as possible can go through the tunnel at anyone time.

You have been commissioned to recommend the maximum and minimum speed limits that allow a good flow of traffic yet maintain safety through the tunnel.

The tunnel is 6.5 kilometres long and will have separate tunnels for each direction of traffic and a third tunnel in case of emergencies.

You need to include the following constraints in your model:

- **The two-second rule**: According to the highway code *"allow at least a two-second gap between you and the vehicle in front on roads carrying fast traffic. The gap should be at least doubled on wet roads and increased still further on icy roads."* (Section 105 from the Highway Code is included)
- Continuous flow of traffic, as one car enters one leaves at the other end of the tunnel

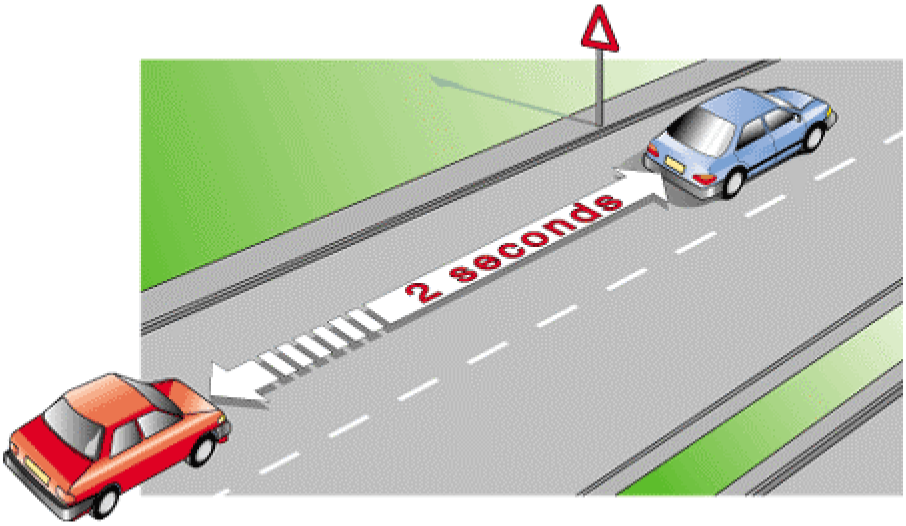
You need to decide on:

- The average length of a car, possibly by research
- Conditions of the road in the tunnel for your model

Stopping Distances from the Highway Code

105: Drive at a speed that will allow you to stop well within the distance you can see to be clear. You should

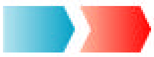
- leave enough space between you and the vehicle in front so that you can pull up safely if it suddenly slows down or stops. The safe rule is never to get closer than the overall stopping distance (see Typical Stopping Distances diagram below)
- allow at least a two-second gap between you and the vehicle in front on roads carrying fast traffic. The gap should be at least doubled on wet roads and increased still further on icy roads
- remember, large vehicles and motorcycles need a greater distance to stop.



Use a fixed point to help measure a two second gap

Typical Stopping Distances

20 MPH



6 metres 6 metres

= 12 metres
(40 feet)
or 3 car lengths

30 MPH

9 metres 14 metres

= 23 metres
(75 feet)
or 6 car lengths

40 MPH

12 metres 24 metres

= 36 metres
(120 feet)
or 9 car lengths

50 MPH

	15 metres	38 metres	= 53 metres (175 feet) or 13 car lengths
60 MPH	18 metres	55 metres	= 73 metres (240 feet) or 18 car lengths
70 MPH	21 metres	75 metres	= 96 metres (315 feet) or 24 car lengths

Thinking Distance

Braking Distance

Average car length = 4 metres

The Long Tunnel Teacher Notes

Assumptions

1. Average car length is 4 metres
2. Tunnel conditions are dry, so two-second rule applies and is obeyed
3. Continuous flow of traffic through the tunnel so as a car enters one leaves
4. Speed is in metres per second

Total length/gap between the back of one car and the back of the next is 4m + “2 second” distance.

“2 second” distance at $x\text{ms}^{-1}$ for seconds is $2x$

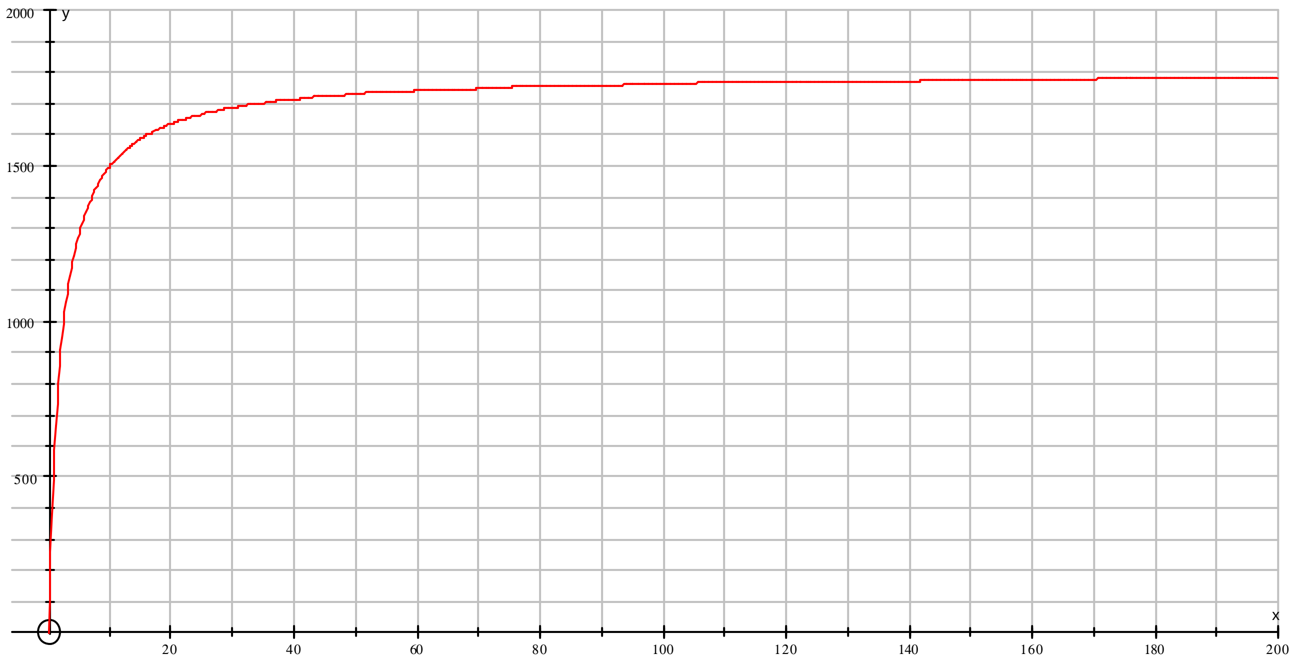
Total length is **4 + 2*x*** metres

Time taken for this car to enter the tunnel is $\frac{4+2x}{x}$ seconds using time = distance/speed

As the tunnel is full of cars it doesn’t matter how long it is so can be modelled by a point. To work out the speed limits examine how many cars can pass this point in an hour.

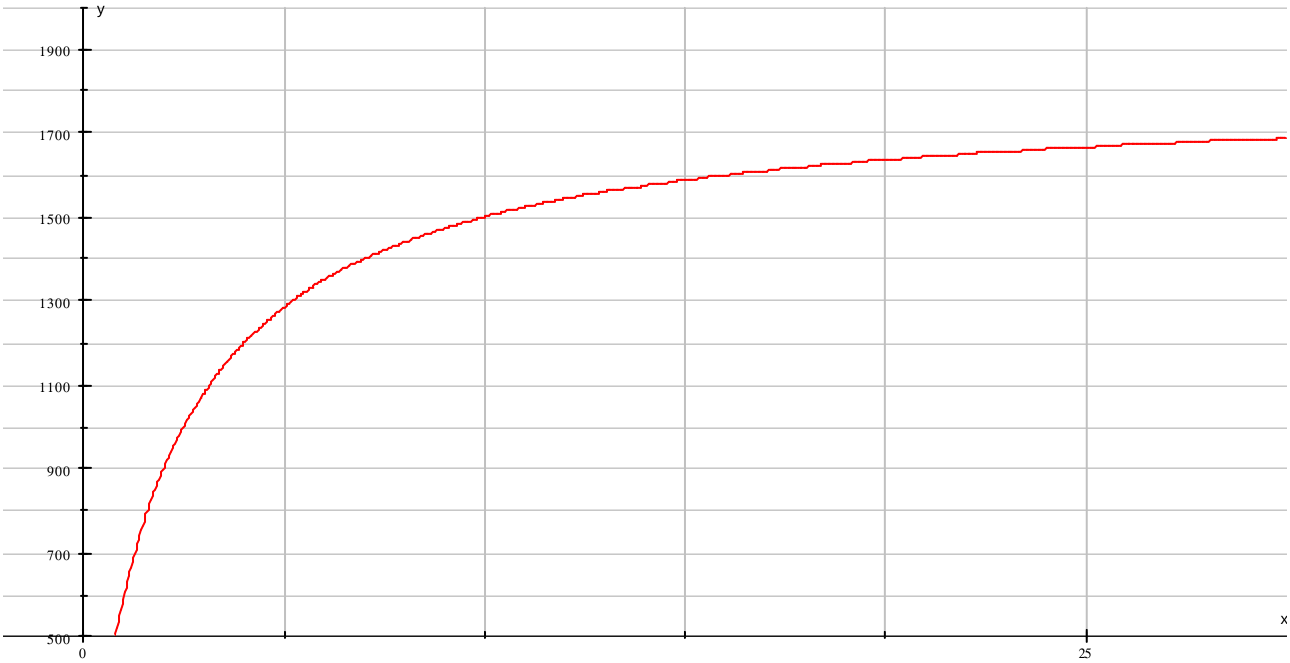
$$\frac{3600}{\frac{4+2x}{x}} = \frac{3600x}{4+2x} = \frac{1800x}{2+x}$$

Graphing $y = \frac{1800x}{2+x}$



There is a section where the graph curves between 10ms^{-1} and 30ms^{-1}

Expanding this area



At $10ms^{-1}$ 1500 cars pass the point, at $20ms^{-1}$ 1636 cars pass the point, at $30ms^{-1}$ 1688 cars pass the point

$10ms^{-1}$ = $36kmh^{-1}$ = 22.5 mph
$20ms^{-1}$ = $72kmh^{-1}$ = 45 mph
$30ms^{-1}$ = $108kmh^{-1}$ = 67.5 mph

Limit should be set somewhere between $72kmh^{-1}$ and $108kmh^{-1}$ as this allows a good amount of cars to pass through without the speed being too excessive.

The students will need to argue their case for their selection. This could be done using a spreadsheet as well.

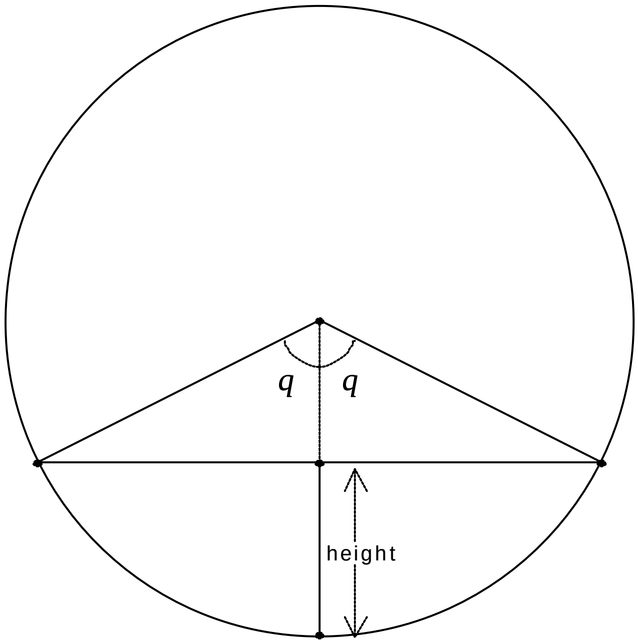
The Milk Tanker

Jonny the driver wants to be able to know how much milk is in his tanker. He thinks that it is possible to design a vertical glass tube that shows the volume of milk left in the tanker.

- What mathematical shape could be used to model the shape of the milk tanker?
- What information do you need to calculate the volume of your mathematical shape?
- Sketch a diagram of the tanker including any dimensions necessary.

Using a cylinder as a model for the main body of the tanker.

- Transfer your dimensions onto a copy of the diagram below.



To help Jonny the driver you need to work out the height he can see in the glass tube and use this to work out the volume.

- Using your dimensions work out the area of the segment, in terms of radius *r* and the angle q.
- Work out the height on the diagram in terms of the radius and the angle q.

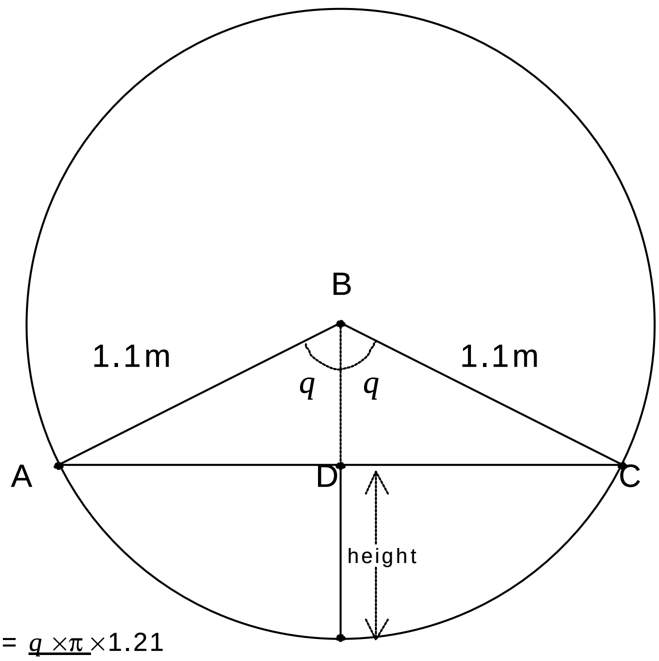
q	**Height**	**Area of segment**	**Volume of milk**
10			
20			
30			

The manager of the company thinks that Jonny's idea is so good that she has vertical glass tubes added to all of the tankers.

Unfortunately none of the tankers are the same dimensions (because it is a mathematics problem) and she wants to have a method of working out the volume of milk from a vertical height for cylindrical tankers of any dimension.

- Using r for the radius, h for the height, q for the angle and l for the length write down the general formula for the volume and height.

Possible solution



Assume the following data
Radius 1.1m, length 10m

Let the length BD be x

Height = $1.1 - x$

Using trigonometry in △ABD

$\cos q = \frac{x}{1.1}$

$\therefore x = 1.1\cos q$

and

$\sin q = \frac{AD}{1.1}$

$\therefore AD = 1.1\sin q$

Area of the sector = $\frac{2q}{360} \times \pi \times 1.1^2 = \frac{q \times \pi}{180} \times 1.21$

Area of △ABC = $^1/_2 \times AC \times x = {}^1/_2 \times (2 \times 1.1\sin q) \times (1.1\cos q)$
$= 1.21\sin q\cos q$

Area of segment = Sector – triangle = $\frac{q \times \pi}{180} \times 1.21 - 1.21\sin q\cos q$

Height = $1.1 - x = 1.1 - 1.1\cos q$

A table of results for half the tanker might look like this:

Angle	cosine	height	Area m2	Volume m3
0	1	0	0	0
10	0.984808	0.016711	0.004263	0.0426265
20	0.939693	0.066338	0.033483	0.3348318
30	0.866025	0.147372	0.109609	1.0960915
40	0.766044	0.257351	0.248931	2.4893067
50	0.642788	0.392934	0.460116	4.6011551
60	0.5	0.55	0.743164	7.4316367
70	0.34202	0.723778	1.089407	10.894074
80	0.173648	0.908987	1.482557	14.825565
90	6.13E-17	1.1	1.900664	19.006636

The volume would depend on the length of the tanker in this case 10m.

How would you find the information if the tanker was more than half full?

To create a table for given heights $\cos q = (1.1 - \text{height})/1.1$

Height	Cosine	Sine	Angle	Area m2	Volume m3
0	1	0	0	0	0
0.1	0.909091	0.416598	24.61998	0.061679	0.61679
0.2	0.818182	0.57496	35.0968	0.171981	1.719813
0.3	0.727273	0.686349	43.34176	0.311325	3.113255
0.4	0.636364	0.771389	50.4788	0.472066	4.720661
0.5	0.545455	0.83814	56.94427	0.649404	6.49404
0.6	0.454545	0.890724	62.96431	0.839813	8.398128
0.7	0.363636	0.931541	68.67631	1.040462	10.40462
0.8	0.272727	0.962091	74.17338	1.248939	12.48939
0.9	0.181818	0.983332	79.52432	1.4631	14.631
1	0.090909	0.995859	84.78409	1.680967	16.80967
1.1	2.02E-16	1	90	1.900664	19.00664

For a general solution

The height
$h = r - r\cos q$

Area of segment $= \frac{q \times \pi}{180} \times r^2 - r^2 \sin q \cos q$

Volume $= l \times \frac{q \times \pi}{180} \times r^2 - r^2 \sin q \cos q$

Assumed previous knowledge:

1. Area of a circle
2. Area of a sector
3. Area of a triangle
4. Know the different parts of a circle including radius, sector, segment
5. Volume of a cylinder
6. Sine and cosine functions in right-angled triangles

This task is designed to be attempted by a group of pupils who have met (and mastered) the six area outlined above.

The task can be attempted individually or collaboratively in small groups.

The actual teaching input at the beginning might be minimal however it is very important that the pupils have an opportunity to discuss their work with a teacher once it has been completed. This discussion will help the pupils develop a clear understanding of the mathematics involved and could include further discussion around the topics involved. For instance explaining that a tabulated answer will solve the problem but mathematicians might like to develop a formula for the volume without the use of the given angle.

Year 9 Able Objectives:

Applying mathematics and solving problems
- Generate fuller solutions to problems.
- Recognise limitations on accuracy of data and measurements; give reasons for choice of presentation, explaining selected features and showing insight into the problem's structure.
- Justify generalisations, arguments or solutions; pose extra constraints and investigate whether particular cases can be generalised further.

Equations, formulae and identities
- Derive and use more complex formulae, and **change the subject of a formula**.

Measure and mensuration
- Know and use the formulae for length of arcs and area of sectors of circles.
- Calculate lengths, areas and volumes in right prisms, including cylinders.
- Begin to use sine, cosine and tangent in right-angled triangles to solve problems in two dimensions.

National Curriculum Attainment Targets:

Ma 1: Using and applying mathematics

Level 8
Pupils develop and follow alternative approaches. They reflect on their own lines of enquiry when exploring mathematical tasks; in doing so they introduce and use a range of mathematical techniques. Pupils convey mathematical meaning through precise and consistent use of symbols that is sustained throughout the work. They examine generalisations or solutions reached in an activity, commenting constructively on the reasoning and logic or the process employed, or the results obtained, and make further progress in the activity as a result.

Exceptional performance
Pupils give reasons for the choices they make when investigating within mathematics itself or when using mathematics to analyse tasks; these reasons explain why particular lines of enquiry or procedures are followed and others rejected. Pupils apply the mathematics they know in familiar and unfamiliar contexts. Pupils use mathematical language and symbols effectively in presenting a convincing reasoned argument. Their reports include mathematical justifications, explaining their solutions to problems involving a number of features or variables.

MA2: Number and algebra

Level 8

Pupils solve problems involving calculating with powers, roots and numbers expressed in standard form, checking for correct order of magnitude. They evaluate algebraic formulae, substituting fractions, decimals and negative numbers. They calculate one variable, given the others, in formulae such as $V = \pi r^2 h$. Pupils manipulate algebraic formulae, equations and expressions, finding common factors and multiplying two linear expressions.

Ma3: Shape, space and measures

Level 8

They use sine, cosine and tangent in right-angled triangles when solving problems in two dimensions.

Exceptional performance

Pupils use sine, cosine and tangent of angles of any size when solving problems in two and three dimensions. They use the conditions for congruent triangles in formal geometric proofs [for example, to prove that the base angles of an isosceles triangle are equal]. They calculate lengths of circular arcs and areas of sectors, and calculate the surface area of cylinders and volumes of cones and spheres.

Resources for Extension for Year 9 Gifted and Talented Pupils

Number & Algebra:

Shell Centre 'Blue Box' – still available from the Shell Centre website!

This concentrates on number patterns leading to algebra and proof.

Skeleton Tower)
Tournament)
Mystic Rose)
Town Hall Tiles)
House of Cards)
Sorting)
Money)
Paper Folding)
Consecutive Sums)
Painted Cubes)
Score Draws)
Frogs)

All of these investigations lead to quadratic or exponential sequences, and build independence

Shell Centre 'Red Box' – still available from the Shell Centre website!

This concentrates on graphs of functions and real-life situations

Looking at exponential functions)
A function with several variables)
Finding functions in situations)
Finding functions in tables of data)
Carbon dating)

All of these lead to quadratic or exponential graphs

Missing Planets – proportionality

Looking at Gradients – real-life graphs of filling bottles and a swimming pool

QCA tasks for the more able – from the QCA website

Sequencing Smartly

Tony Gardiner's book Maths Challenge 1

Proof – building up to proof

True, False and Iffy – starts with number facts but could extend to include many topics or even get the pupils to invent some of their own

Define a number

Shape, space and measure:

Shell Centre 'Red Box' – These are all investigations that lead to graphs

The Journey	)	These are all speed-
Are graphs just pictures	)	distance-time graphs
Sketching graphs from pictures	)	and often include gradient
Point of no return	)	calculations and processing
Ffestiniog Railway	)	information

Max Box	)	
Design a can	)	Optimisation problems leading to graphs or
Camping	)	spreadsheet work

The Cassette Tape – Circumference and graph plotting

Finding functions in situations - has a lovely extension to the angles in a regular polygon investigation.

Sketching graphs from pictures – sketching paths of particles (an excellent extension to work on loci)

The Harbour Tide – interpreting a trigonometric graph

Tony Gardiner's book Maths Challenge 1

Bigger or Smaller – similar triangles

QCA tasks for the more able – from the QCA website

Awkward Areas

Proving Pythagoras

Beyond Pythagoras – based on the GCSE coursework task

Areas of Regular Polygons - based on the GCSE coursework task

Handling Data:

QCA tasks for the more able – from the QCA website

Everyone's a Winner – probability

Web addresses

Shell Centre Blue Box: http://www.mathshell.com/scp/ppn51.htm
Shell Centre Red Box: http://www.mathshell.com/scp/lfg50.htm

QCA tasks for the more able index: http://www.qca.org.uk/2624_6934.html

Sequencing smartly Combined Tasks:
http://www.qca.org.uk/downloads/ks3_maths_combined_sequencing_tasks.pdf
Sequencing smartly – Pupil work sheet:
http://www.qca.org.uk/downloads/4974_ma_above_seq_ps1.pdf
Sequencing smartly – Solutions and what to look for:
http://www.qca.org.uk/downloads/4975_ks3_ma_above_seq_sol.pdf

Awkward Areas Combined Tasks:
http://www.qca.org.uk/downloads/ks3_maths_combined_awkward_areas_tasks.pdf

Awkward Areas – Pupil Sheet:
http://www.qca.org.uk/downloads/4976_ks3_ma_awk-areas.pdf
Awkward Areas – Solutions and what to look for:
http://www.qca.org.uk/downloads/4977_ma_above_awk_sol.pdf

Proving Pythagoras Combined tasks:
http://www.qca.org.uk/downloads/ks3_maths_combined_pythagoras_tasks.pdf
Proving Pythagoras – Pupil Sheet:
http://www.qca.org.uk/downloads/ks3_ma_above_04_pythagoras_task.pdf
Proving Pythagoras – Solutions and what to look for:
http://www.qca.org.uk/downloads/6935_ks3_ma_above_04_pythag_sols.pdf

Puzzling Probability Combined tasks:
http://www.qca.org.uk/downloads/ks3_maths_combined_probability_tasks.pdf
Puzzling Probability – Pupil Sheet:
http://www.qca.org.uk/downloads/ks3_ma_above_04_probability_task.pdf
Puzzling Probability - Solutions and what to look for:
http://www.qca.org.uk/downloads/ks3_ma_above_04_probability_sols.pdf